2077/01/1

For Grade XI:

(1)

Curve Sketching (2+4)

Priliminaries :- (OR Curve tracing)

Domain and range :-

If $y = f(x)$ be a function, the set of all x for which y is well-defined is the domain of the function and the set of all values of y consitute the range of the function.

For example: The domain of the function $y = \frac{1}{x}$ is $\mathbb{R} - \{0\}$ because it is defined for all values of x except 0. For the range, we write the eqⁿ as $x = \frac{1}{y}$. So, y can take any value other than 0. This means the range of the function is $\mathbb{R} - \{0\}$.

Even and odd functions :-

For a function $y = f(x)$ to be even, we must have $f(x) = f(-x)$, for all x.

For a function $y = f(x)$ to be odd, we should have $f(x) = -f(-x)$, for all x.

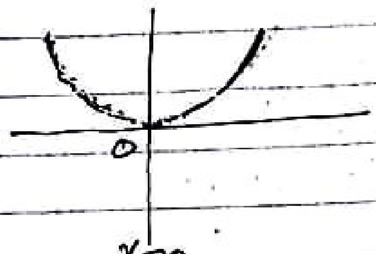
Example 1, The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$, whereas the function $f(x) = x^3$ is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Exercise: Check whether the following functions are odd or even.

1. $y = f(x) = x\sin x + \cos x$, (even)
2. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (odd)
3. $y = \sqrt{1+x} - \sqrt{1-x}$ (odd)

3. Symmetry: When a transformation uses a line that acts as a mirror, with an image reflected in the line, we say the curve (or function) is symmetrical about that line.

Example: $y = x^2$ is symmetrical about a line $x = 0$ (y-axis)



$x=0$

[The figure is identical to the right and left of the line $x = 0$]

4. Periodic function:

A function $y = f(x)$ is said to be periodic if $f(x+p) = f(x)$ for all x.

The smallest value of p is called the period of the function.

For Example $f(x) = \sin x$ has period 2π, because

$$f(x+2\pi) = \sin(2\pi + x) = \sin x = f(x), \text{ for all } x.$$

Note that: Functions $\sin ax$, $\cos ax$ have periods $\frac{2\pi}{|a|}$

whereas $\tan ax$ has period $\frac{\pi}{|a|}$

Exercise: Test the periodicity of $f(x) = \cos x + \sin x$

Solution :- Here, $f(2\pi + x) = \cos(2\pi + x) + \sin(2\pi + x)$

$= \cos x + \sin x$

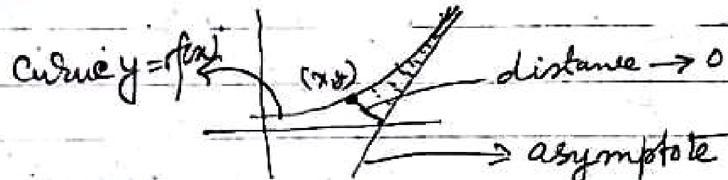
$= f(x)$

and clearly, 2π is the smallest no. such that $f(2\pi + x) = f(x)$. Hence, the given function has period 2π.

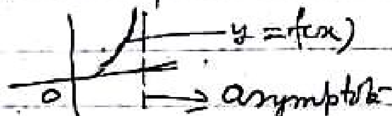
More Exercise : 1) $y = f(x) = \sin 2x$ 2) $y = f(x) = \tan\frac{x}{4}$

Asymptotes :-

A Straight line is said to be an asymptote to the curve $y = f(x)$ if the distance of the line from any point (x, y) tends to 0 as $x \to \infty$ or $y \to \infty$.

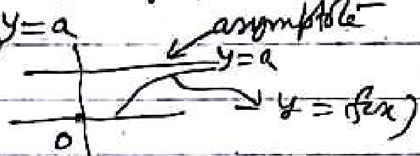


For Vertical asymptote $x = a$, $y = f(x) \to \infty$, as $x \to a$



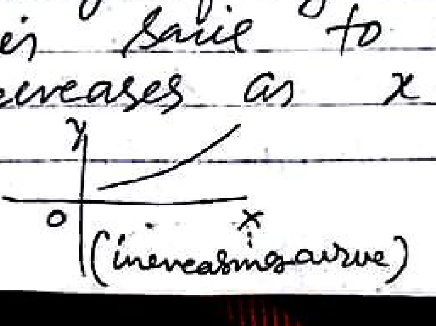
For horizontal asymptote, $y = a$

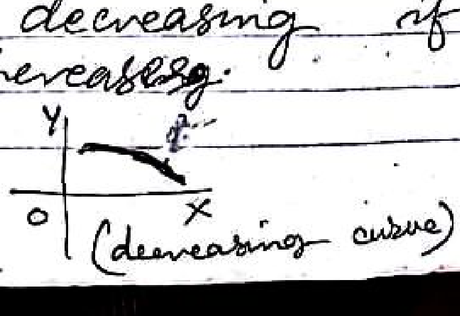
$y \to a$, as $x \to \infty$



Increasing and decreasing functions

The function $y = f(x)$ is said to be increasing if y increases as x increases and is said to be decreasing if y decreases as x increases.





Guidelines for Sketching Curves (or tracing of curves).

The following information regarding the eq^n should be determined.
(Characteristics of graph)

(1.) **Domain & Range**
Find the domain of $y=f(x)$. It is the set of all x for which y is defined.
Write the range, (if necessary).

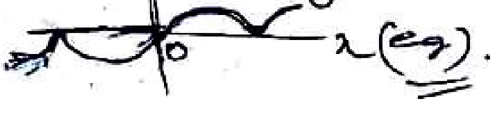
(2) **Origin**:
Decide whether the curve passes through the origin or not. If the equation does not contain a constant term, it passes through the origin. (For algebraic eq^ns)
e.g. $y=x^2$ passes through the origin whereas $y=2x+3$ does not pass through the origin
$\boxed{\text{Put } x=0,\ y=0 \text{ in the eq}^n}$. If it satisfies (eq^n), the curve passes through the origin).

(3.) **Symmetry**:
Check the symmetry of the curve about x-axis, y-axis, about origin etc

Replacing x by $-x$, if the equation remains unchanged, then the curve is symmetrical about y-axis (i.e. curve is identical to the right and left of y-axis)

Replacing y by $-y$, if the equation remains unchanged, then the curve is symmetrical about x-axis (i.e the graph is identical above and below x-axis)

$\boxed{\text{Symmetry about origin}}$
If the graph of $f(x)$ coincides (overlaps) after rotating it through 180°, the curve is said to be symmetrical about origin.



Replacing x by $-x$ and y by $-y$, if the equation remains unaltered, then the curve is symmetrical about the origin.

Note: Even function is symmetrical about y-axis & odd → about origin

Ex: 1. $y = x^2$ is symmetrical about y-axis because, if we replace x by $-x$, the eqⁿ becomes $y = (-x)^2 = x^2$, which remains unaltered.

2. $y = 2x+3$ is not symmetrical about x-axis and y-axis.

3. $y^2 = 4ax$ is symmetrical about x-axis because if we replace y by $-y$, the eqⁿ becomes

$$(-y)^2 = 4ax \quad \text{or} \quad y^2 = 4ax,$$ which remains unaltered.

4. $x^2+y^2=1$ is symmetrical about origin because replacing x by $-x$ and y by $-y$, the equation remains unaltered.

(4) Points (Noticiable points) where the curve meet x-axis and y-axis

Find the points where the curve meet x-axis and y-axis (if any)
Put $x=0$ and obtain y to find the points on y-axis
Put $y=0$ and find x, to find the points on x-axis.
Other noticeable points should be noted (if any)

(5.) Table showing the variation of y as x

Prepare a table that shows the variation of y as x as follows:

x	.	.	.	.	.
y	.	.	.	.	.

6. Increasing / decreasing.

Find the intervals in which $y=f(x)$ is increasing or decreasing (if they exist)

$\frac{dy}{dx} > 0$ in $a \le x \le b \Rightarrow (y=f(x))$ is increasing

$\frac{dy}{dx} < 0$ in $a \le x \le b \Rightarrow (y=f(x))$ is decreasing

7. Maxima and minima

Find the points of maxima and minima (if any).

8. Asymptotes

Determine asymptotes (if any)
(Bounded curve has no asymptotes).

9. Periodicity.

Find the period of the function (if exists).

Example:- Period of $\sin ax$ is $\frac{2\pi}{|a|}$

(Period of $\cos ax$) $= \frac{2\pi}{|a|}$

(Period of $\tan ax$) $= \frac{\pi}{|a|}$ □

Note. 1. All Above mentioned points (1-9) are generally required to trace (sketch) any curve. However, it is not necessary to note all the points for all curves.

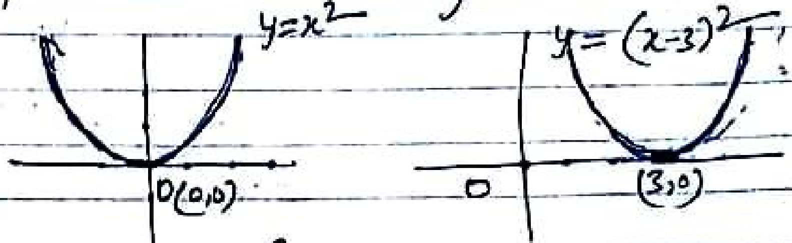
2. $y = x^2$ is symmetrical about $x=0$ (i.e. y-axis) and so, $y=(x-3)^2$ is symmetrical about the line $x-3=0$ i.e. $x=3$.

Further Guidelines (Transformations)

Shifting (or translation) of graphs

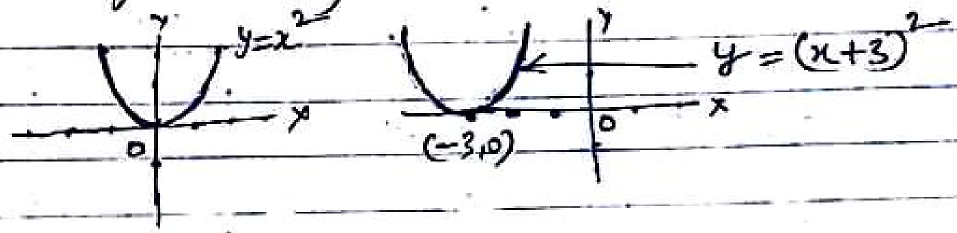
1. The graph of $y=f(x-a), a>0$ is as that of $y=f(x)$. Shifted by 'a' units to the right.

e.g.



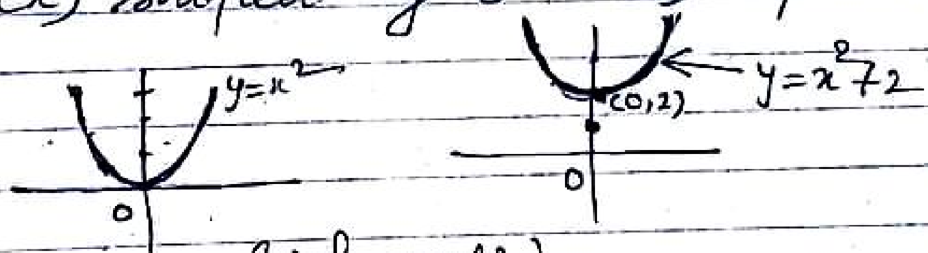
2. The graph of $y=f(x+a)$, $a>0$ is as that of $y=f(x)$ shifted by 'a' units to the left.

e.g.



3. The graph of $y=f(x)+b$ $(b>0)$ is as that of $y=f(x)$ shifted by 'b' units upward.

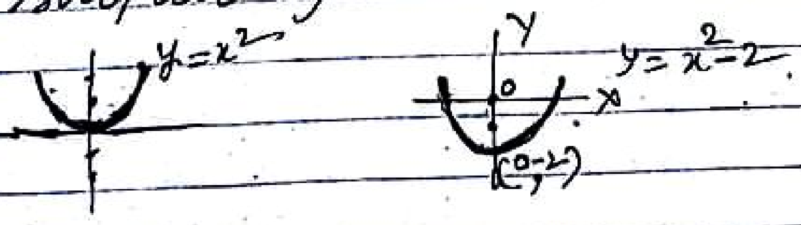
e.g.



(we can use a single graph)

4. The graph of $y=f(x)-b$ $(b>0)$ is as that of $y=f(x)$ shifted by 'b' units downwards.

e.g.

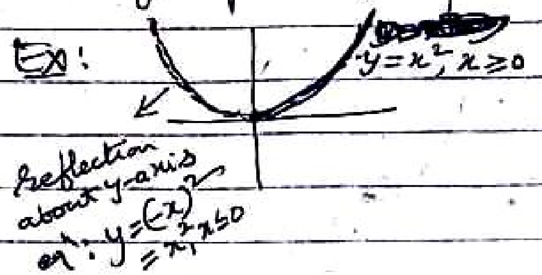


Exercise:- Do

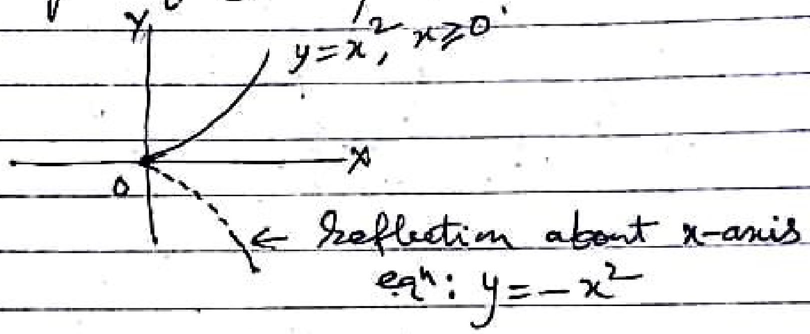
(Reflection of graphs)

1. The graph of $y = f(-x)$ is the reflection of the graph of $f(x) = y$ about y-axis.

Ex:



2. The graph of $y = -[f(x)]$ is the reflection of the graph of $y = f(x)$ about x-axis.



Note:

1. While sketching $y = ax^2 + bx + c$, convert the equation into the form

$$y = a(x \pm h)^2 \pm k, \quad h, k > 0$$

When $a > 0$, the graph is concave upward

when $a < 0$, the graph is concave downward

2. $(x-h)^2 = 4a(y-k)$ is a parabola having vertex (h,k), $a > 0$. Symmetric about $x = h$ → U (Graph)

3. $(y-k)^2 = 4a(x-h)$, $a > 0$ " " → C (graph)
Symmetric about $y = k$

Graphs of some well-known curves

(5)

Ex-(1) Sketch the graph of $\boxed{y = f(x) = 3x+2}$

Solution:- The following information regarding the equation is determined.

(1-) Domain
The domain of $f(x)$ is $(-\infty, \infty)$
and range $= (-\infty, \infty)$

(2.) It does not pass through the origin, as the constant term occur in the equation.

(3.) If $y = 0$, $x = -\frac{2}{3}$. So the curve meets x-axis at $(-\frac{2}{3}, 0)$.

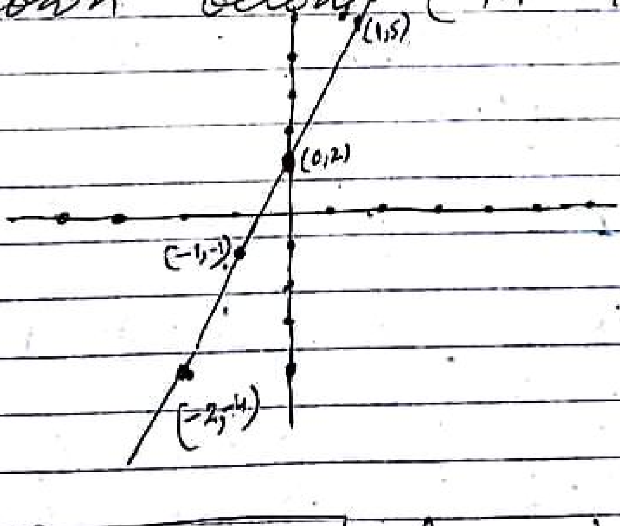
If $x = 0$, $y = 2$. The curves cuts the y-axis at point $(0,2)$

(4.) Table for variation of y as x

x	-2	-1	$-2/3$	0	1
y	-4	-1	0	2	5

(5) y increases as x increases.

Thus, the graph of the function is as shown below (in the graph paper)



Exercise: $\boxed{y = 3x-2}$, hence draw the graph of

$y = \boxed{3(x-2)-2+1} = 3x-7$ (its graph is the graph of $y = 3x-2$, shifted 2 units to the right and shifted above by 1)

★ After relflection about x-axis
$(x, y) \longrightarrow (x, -y)$

★ After reflection about y-axis
$(x, y) \longrightarrow (-x, y)$.

2. Sketch the graph of $\boxed{y = x^2}$

Solution :-
The following information regarding the equation is noted.

(1) Domain / Range
Domain $= (-\infty, \infty)$
Range $= [0, \infty)$ (not required)

(2) Origin :
The given curve passes through the origin, since if $x = 0, y = 0$, $0 = 0^2$ i.e. $0 = 0$)

~~(3) Points where the curve meet x-axis/y-axis~~

(3) Symmetry :
The curve is symmetrical about y-axis
($\because$ If we replace x by $-x$, the eqⁿ remains unchanged.)

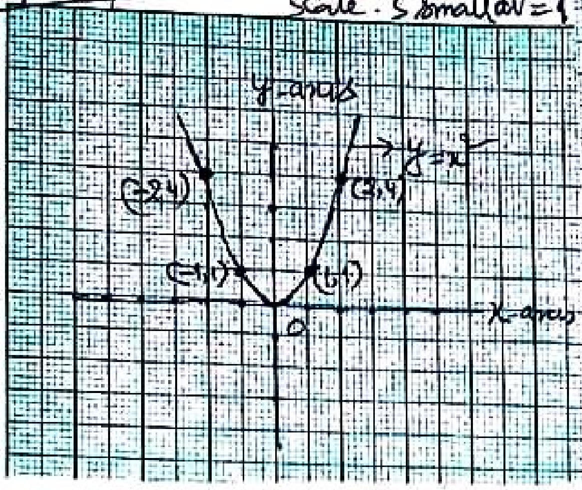
(4) Table for the variation of y as x

x	0	1	2	-1	-2
y	0	1	4	1	4

(5) As x increases, y also increases.

(6) Vertex (0,0). Thus, the graph of the function is as shown in the graph paper.

graph of $y=x^2$ Scale : 5 small div = 1



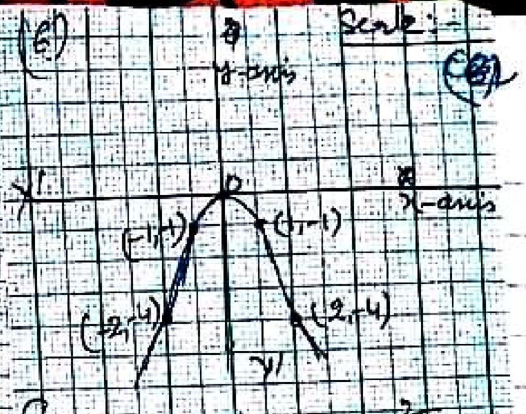
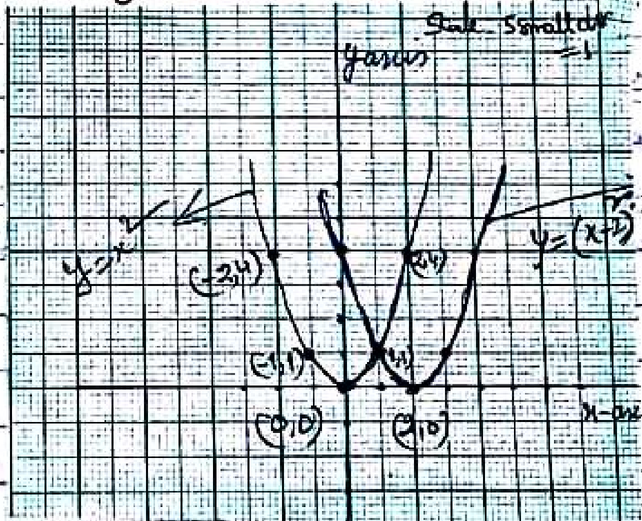


fig: graph of $y=-x^2$
Reflection of $y=x^2$ about x-axis

3. Sketch the graph of $y=x^2$ and hence sketch the graph of $y=(x-2)^2$.

Soln:- Write characteristics as in Example 2. Then

note that, the graph of $y=(x-2)^2$ is as the graph of $y=x^2$, shifted 2 units to the right. The graphs of $y=x^2$ and $y=(x-2)^2$ are shown on the graph paper.



Take:
Scale, 10 small div = 1
in Exams.

4 Sketch the graph of $y=x^2$ and hence sketch the graph of $y=x^2-4x+3$

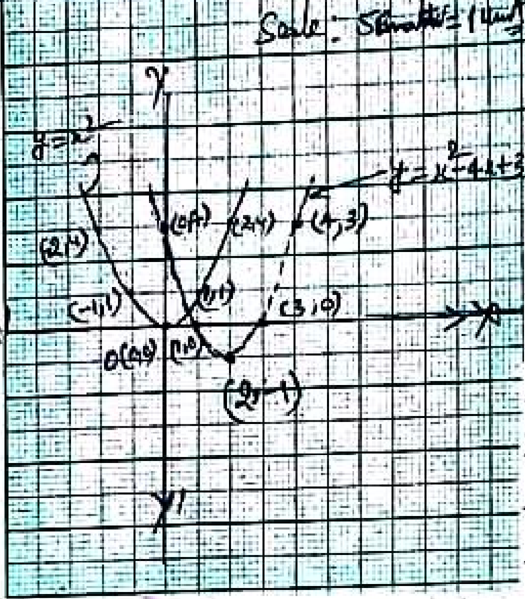
Soln:- Write characteristics as in Example 2, for $y=x^2$.
Now, we can write $y=x^2-4x+3$ as

$$y=(x-2)^2-4+3$$

$$\text{ie } y=(x-2)^2-1$$

So, the graph of $y = x^2 - 4x + 3$ is the graph of $y = x^2$ shifted by 2 units to the right and shifted below by 1 unit.

Hence, the graphs of both $y = x^2$ and $y = x^2 - 4x + 3$ are as shown in the graph paper.



(Take) [In Exams: Scale: 10 small div = 1]

Exercise 1: Sketch the graph of $\boxed{y = x^2}$ and hence sketch the graph of $y = (x-3)^2$.

Exercise 2: Sketch the graph of $y = x^2$ and hence sketch the graph of $y = x^2 - 6x + 5$.

(5.) Sketch the graph of $y = x^2$ and hence sketch the graph of $y = -(x^2 - 4x + 3)$
$= -x^2 + 4x - 3$

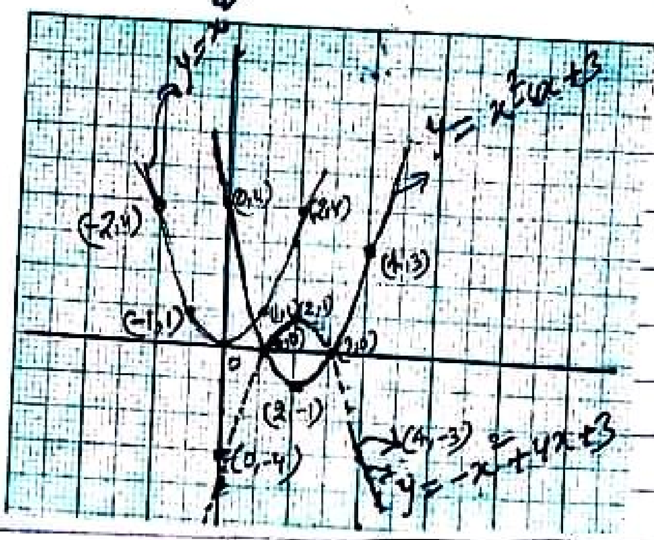
(X)

Sol^n: For $y = x^2$ do as in Example 2.

The graph of $y = x^2 - 4x + 3$ is the graph of $y = x^2$ shifted right by 2 units and below by 1 unit.

And, the graph of $y = -x^2 + 4x - 3$ is the reflection of the graph of $y = x^2 - 4x + 3$ about x-axis.

thus the graphs of $y = x^2$, $y = x^2 - 4x + 3$ and $y = -x^2 + 4x - 3$ are as shown in the graph paper.



6. Sketch the curve $\boxed{y = x^2 + 2x + 3}$ describing its characteristics.

Solⁿ: Characteristics of the graph of the given function are as given below.

$$y = x^2 + 2x + 3$$
$$= (x+1)^2 - 1 + 3$$
$$= (x+1)^2 + 2 \quad (\geq 2)$$

1. **Domain / range**

$$D(f) = (-\infty, \infty), \quad \text{domain}$$
$$R(f) = [2, \infty) \quad ; \quad \text{range}$$

2. **Origin**:
The curve does not pass through the origin, as the eqⁿ contains a constant term.

3. **Points on Axes**.
The curve meets y-axis at the point (0,3) (Putting $x=0$ in the eqⁿ, we get $y=3$)

~~does not meet x-axis because if y=0, $x^2+2x+3=0$, which gives no real value of x, $x = \frac{-2 \pm \sqrt{4-12}}{2}$ — negative~~

$x \to -x \Rightarrow$ eqn, no change

and it passes through $(-1, 2)$ $(\because y = (x+1)^2+2)$

4. **Symmetry** :-

(Vertex)

The given eqn is $y = (x+1)^2 + 2$, which is in the

form of $y = x^2 + 2$.

$\therefore$ the curve is symmetrical about the line $x = 0$ (Y-axis)., i.e. $x + 1 = 0$

i.e. $x = -1$

5) **Table** for the variation of y as x

x	-3	-2	-1	0	1	2
y	$+6$	3	2	3	8	11

Rough

$y = (x+1)^2 + 2$

6) **dy/dx** (first derivative)

$$\frac{dy}{dx} = 2x + 2$$
$$= 2(x+1)$$

$\therefore \frac{dy}{dx} > 0$, for $x > -1$

$\Rightarrow \therefore$ the curve is increasing for $x > -1$

~~(decreasing not required, because it is symmetrical about $x = -1$)~~

Rough

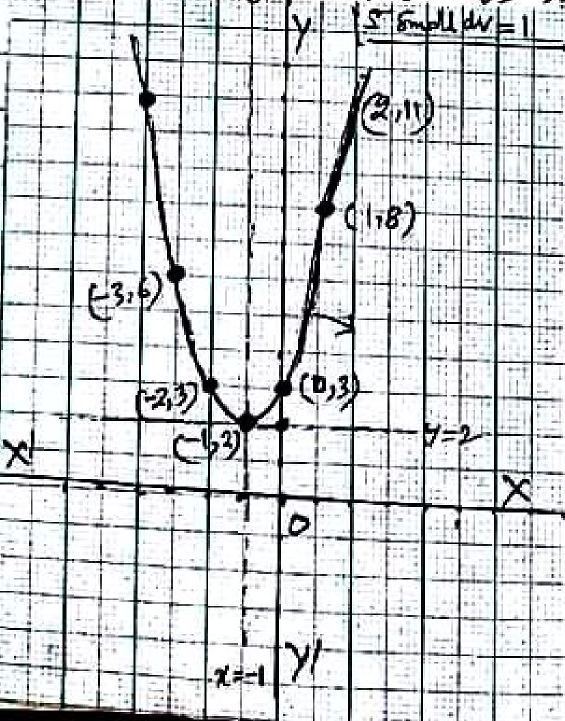
$\frac{dy}{dx} > 0$

$\Rightarrow 2(x+1) > 0$

$\Rightarrow x + 1 > 0$

$\Rightarrow x > -1$

Thus, the graph of the given function is as shown in the graph paper.



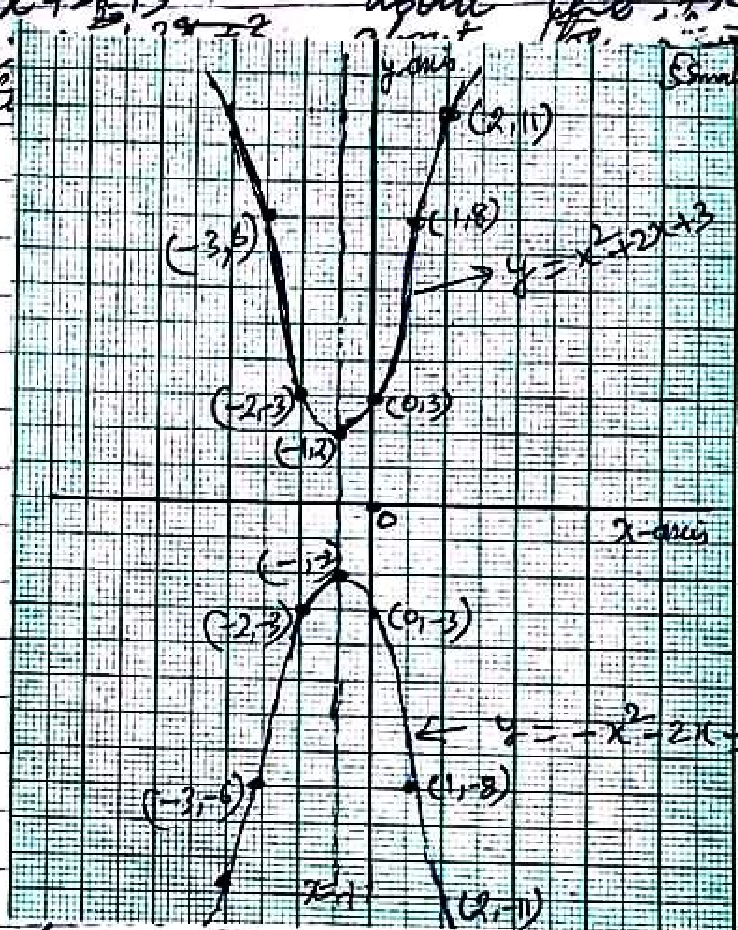
Exercise

$y = x^2 - 4x + 3$

$y = 3x^2 + 4$

Note:- The graph of $y = -(x^2+2x+3)$

i.e. $y = -x^2-2x-3$ is the reflection of the graph of $y = x^2+2x+3$ about the x-axis, as shown in



$\boxed{\text{H.W: } y = -x^2+4x-3}$

Example 7 :- Sketch the graph of $y = -x^2+4x-3$.

Solⁿ:- Firstly, we draw the graph of $y = x^2-4x+3$ (-ve of given y)

The following information regarding the equation $y = x^2-4x+3$ is determined.

Here,

$$y = (x-2)^2-4+3$$
$$= (x-2)^2-1$$

1. Domain / Range

$$\text{Domain} = (-\infty, \infty)$$
$$\text{Range} = [-1, \infty)$$

2. Origin : The curve does not pass through the origin.

$\boxed{y=(x-2)^2-1}$

3. Noticiable points
The curve meets the y-axis at (0,3) and passes through the point (2,-1). Meets x-axis at points (1,0) & (3,0). (Put y=0)

4. Symmetry :- The given equation is $y=(x-2)^2-1$ which is of the form $Y^2=X^2-1$. Therefore it is symmetrical about Y-axis (ie. X=0, ie. x-2=0, ie x=2).

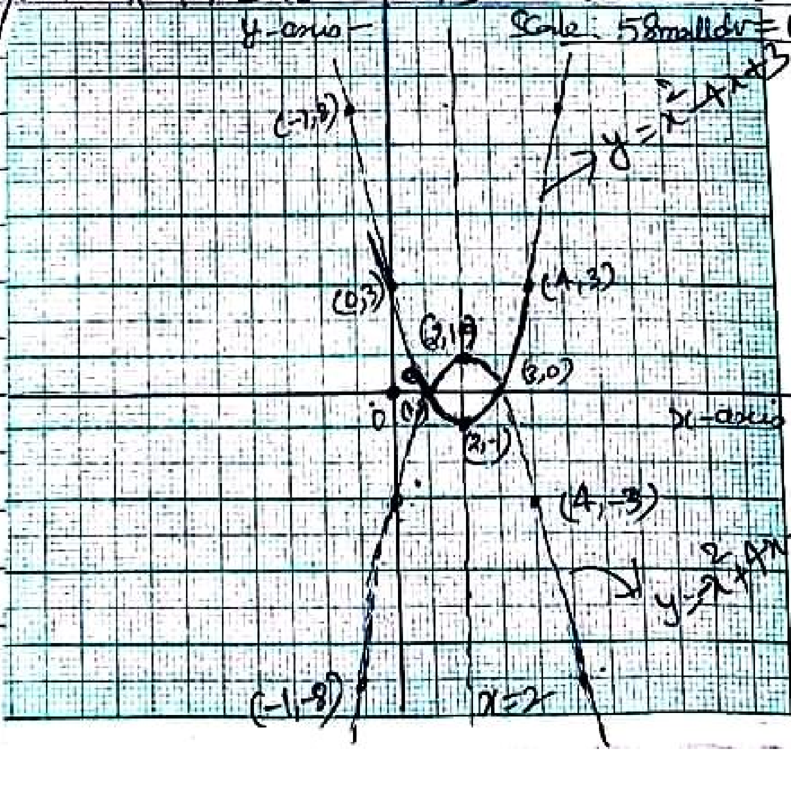
5. Table for the variation of y as x

x	-2	-1	0	1	2	3	4
y	+15	8	3	0	-1	0	3

6. $\frac{dy}{dx} = 2x-4 = 2(x-2)$

Thus $\frac{dy}{dx} > 0$, for $x>2$. This means the graph is increasing for $x>2$.

Keeping in mind the information above mentioned, we sketch the graph ~~[illegible]~~ of $y=x^2-4x+3$. The graph of $y=-x^2+4x-3$ is the reflection of graph of $y=x^2-4x+3$ about x-axis, as shown below (aside)



Example 8 :- Sketch the curve $\boxed{y = x^3}$ and hence sketch $y = (x-3)^3$.

Solution :- The following information regarding the eqn $y = x^3$ is obtained.

(1.) **Domain** $= (-\infty, \infty)$
Range $= (-\infty, \infty)$

(2.) **Origin** :- The curve passes through the origin (since there is no constant term in the equation)

(3.) **Symmetry** :- Not symmetrical about x & y axis but symmetrical about origin
$(\because -y = (-x)^3 \Rightarrow y = x^3)$

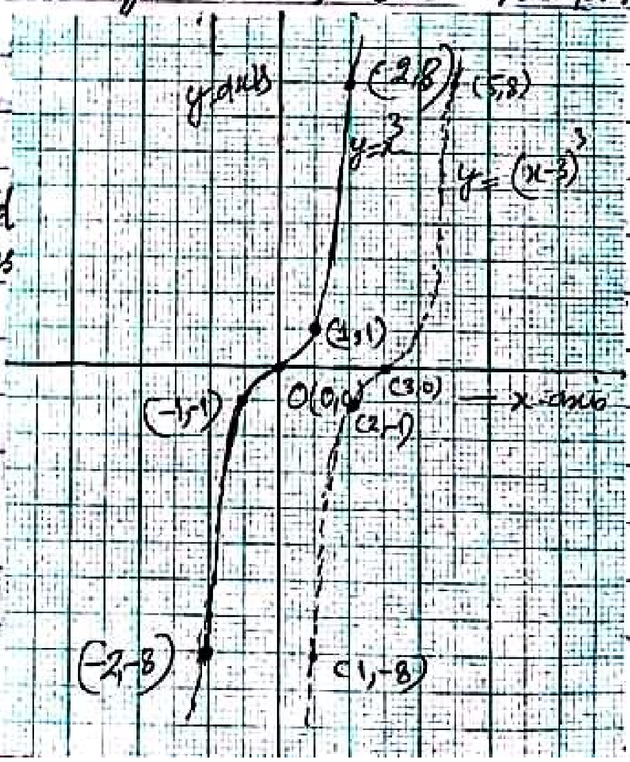
(4.) Table showing the variation of y as x

x	0	1	2	~~3~~	-1	-2
y	0	1	8	~~27~~	-1	-8

5.) First derivative

$$\frac{dy}{dx} = 3x^2 \geq 0 \text{ (always)}$$

$\therefore$ the curve is increasing. Thus, the graph is as under...



The graph of $y = (x-3)^3$ is the graph of $y = x^3$ shifted 3 units to the right, as shown in the graph.

Exercise :

$y = (x-2)^3 + 3$

$y = (x-3)^3 - 1$

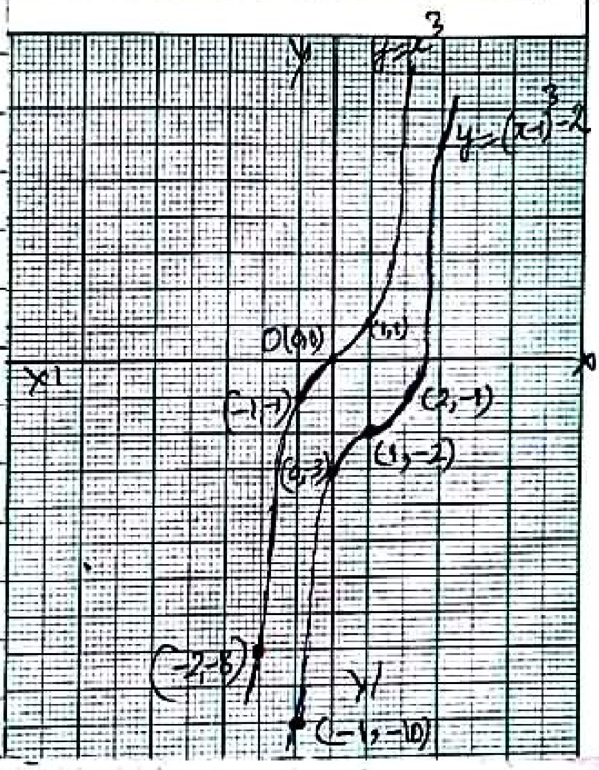
$y = (x+2)^3$

$\boxed{\text{Example : 9}}$: Sketch the graph of $\boxed{y = x^3}$ and hence draw the graph of $y = (x-1)^3 - 2$.

Solution :- The characteristics of $y = x^3$ are as in Example 8.

The graph of $y = (x-1)^3 - 2$ is as that of $y = x^3$ shifted 1 unit to the right and shifted below by 2 units.

The graphs of both $y = x^3$ and $y = (x-1)^3 - 2$ are shown in the graph paper.



$\boxed{\text{Example 10}}$

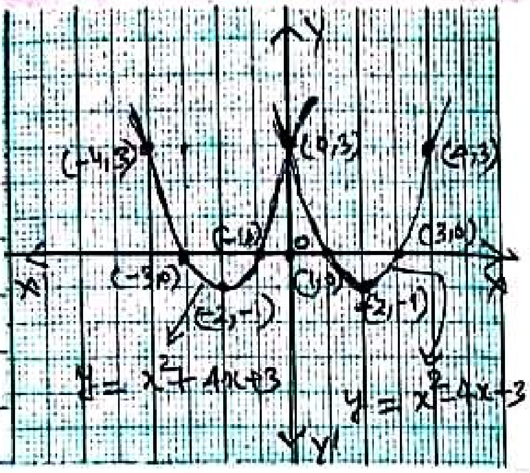
Draw the graph of $y = f(x) = x^2 - 4x + 3$ and hence sketch the graph of $y = f(-x) = x^2 + 4x + 3$.

Solution :-

For the graph of $y = x^2 - 4x + 3$, see Example 7.

The graph of $y = f(-x) = x^2 + 4x + 3$ is ~~as that of~~ the reflection of the graph of $y = x^2 - 4x + 3$ about y-axis, as shown on the graph paper.

$(x, y) \xrightarrow{\text{Note}} (-x, y) \rightarrow$ reflection about y-axis



$\boxed{\text{Example 11}}$ Sketch the graph of $\boxed{y = x - x^2}$ and hence draw the graph of $y = -x + x^2$.

<u>Sol</u>

Solⁿ:- The given eqⁿ is

$$y = x - x^2$$
$$= -[x^2 - x]$$
$$= -\left[\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right]$$

i.e. $y = -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{1}{4} = \tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2$

The following information regarding the eqⁿ is noted.

(1.) Domain / Range

Domain $= (-\infty, \infty)$

Range $= (-\infty, \tfrac{1}{4}]$

(2.) Origin : The curve passes through the origin.

(3.) Symmetry : It is not Symmetrical about x-axis and y-axis. and Symmetrical about line $x = \tfrac{1}{2}$.

(4) Noticeable points

The graph meets x-axis at points (0,0) (1,0) (Put $y=0$)

~~The graph meets y-axis at (x, …~~

It passes through $\left(\tfrac{1}{2}, \tfrac{1}{4}\right)$, (vertex)

(5) Table for variation of y as x

x	-2	-1	0	1	2	3
y	-6	-2	0	0	-2	-6

(6) $\frac{dy}{dx} = 1 - 2x$

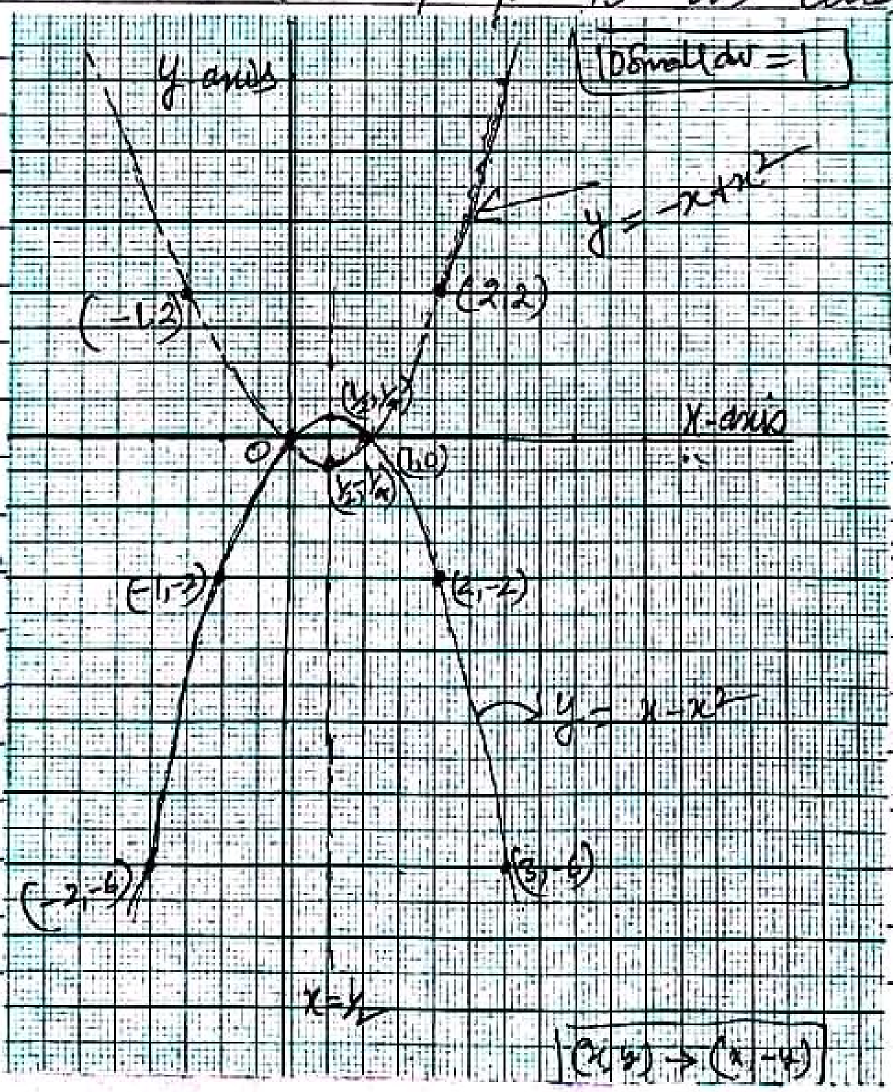
$\therefore \frac{dy}{dx} > 0$ for $1 - 2x > 0$

i.e. $2x < 1$

i.e. $x < \frac{1}{2}$

$\therefore$ the curve is increasing for $x < \frac{1}{2}$.

Since $y = -x + x^2 = -(x - x^2)$, $\therefore$ its graph is the reflection of the graph of $y = x - x^2$ about x-axis. The graph is as under.



Example 12

Draw the graph of $\boxed{y = f(x) = (x-1)(x-2)(x-3)}$

Solution: - **Characteristics**

(1) domain / Range

domain $= (-\infty, \infty)$
range $= (-\infty, \infty)$

→ does not pass through (0,0)
→ not symmetrical
↓
Not required

(2) Plotticable points

when $y = 0$, $x = 1$ or 2 or 3. Therefore, the curve meets x-axis at points $(1,0)$, $(2,0)$ & $(3,0)$.
When $x = 0$, $y = -6$. Therefore curve meets y-axis at $(0,-6)$

(3) Sign of y:

Range	$x-1$	$x-2$	$x-3$	$y=(x-1)(x-2)(x-3)$
$-\infty$ to 1	$-$	$-$	$-$	$-$
1 to 2	$+$	$-$	$-$	$+$
2 to 3	$+$	$+$	$-$	$-$
3 to ∞	$+$	$+$	$+$	$+$

$\therefore$ when $x < 1$, $y < 0$ and decreases to $-\infty$ as x decreases to $-\infty$

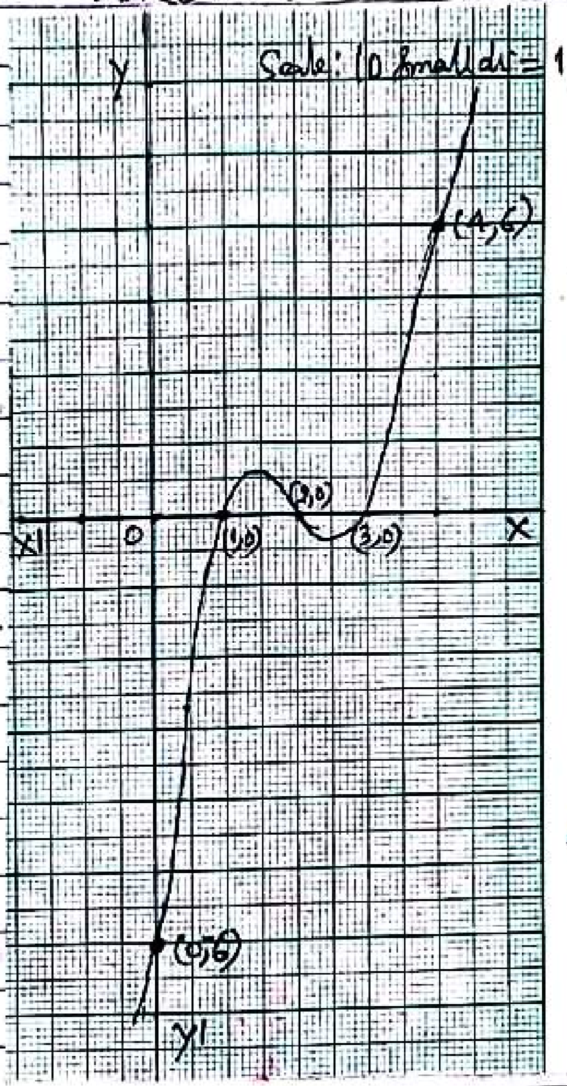
when $1 < x < 2$, $y > 0$ and so, the part of the curve lies above x-axis.

when $2 < x < 3$, $y < 0$ and so, the part of the curve lies below x-axis.
When $x > 3$, $y > 0$ and increases to ∞ as x increases to ∞.

(4) Table for variation of y as x.

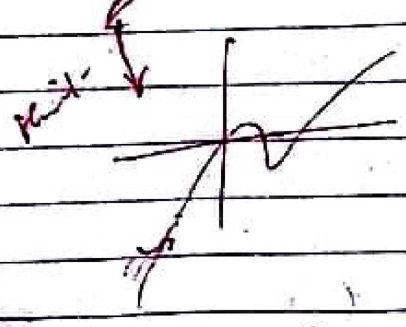
x	-1	0	1	2	3	4
y	-24	-6	0	0	0	6

Keeping in mind the above mentioned information, we sketch the curve in the graph paper as



Exercise :- $y = (x-0)(x-1)(x-2)$ i.e. $y = x(x-1)(x-2)$

$y = (x-2)(x-3)(x-4)$



Example 13 Draw the graph of $y = 1 \cdot \sin x$ describing its characteristics.

Solution:— The following information regarding the equation is obtained.

(1) Domain / Range

$$\text{Domain} = (-\infty, \infty)$$
$$\text{Range} = [-1, 1]$$

(2) Origin:
When $x = 0$, $y = \sin 0 = 0$. Thus the curve passes through the origin.

(3) Symmetry:
It is not symmetrical about x-axis and y-axis, symmetric about origin (After $x \to -x$, $y \to -y$, eqn unchanged)

(4) Noticeable points
When $y = 0$, $\sin x = 0$ and so,
$x = n\pi + (-1)^n \cdot 0$ $(n = 0, \pm 1, \pm 2, \dots)$. thus,
the graph cuts x-axis at $x = \dots, -2\pi, -\pi, 0, \pi, 2\pi \dots$

(5) Period
period $= 2\pi$

∴ Period of $\sin ax$ is $\frac{2\pi}{|a|}$

(6) Maxima / minima

Maximum value is 1, which occur at
$x = \dots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \dots$

Rough
$y = 1 \cdot \sin x = 1$
$\Rightarrow \sin x = 1$
$\Rightarrow x = n\pi \pm (-1)^n \frac{\pi}{2}$
$n = 0, \pm 1, \pm 2, \dots$
$+2\pi$

(7) Minimum value is -1, which occur at
$x = -\frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

$\sin x = -1$
$x = \frac{3\pi}{2}, -\frac{\pi}{2}$

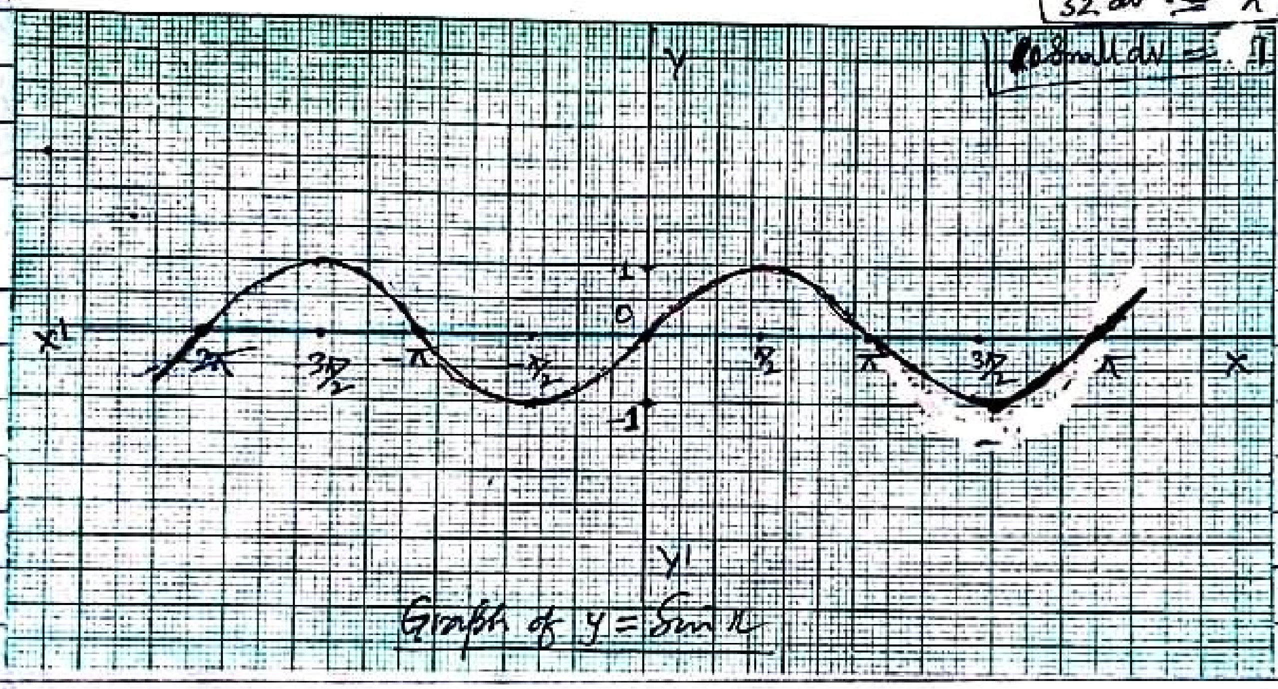
(table ← not required)

$\frac{dy}{dx} = \cos x$ $\quad$ π — $\frac{3\pi}{2}$ — 2π

>0 for 0 to $\frac{\pi}{2}$ etc. <0 for $\frac{\pi}{2}$ to $\frac{3\pi}{2}$

(7.) $y = \sin x$ increases from 0 to $\pi/2$ and then decreases from $\pi/2$ to π etc.

Thus the graph of the given function is as shown in the graph.



Graph of $y = \sin x$

Note :- To trace $y = 3\sin x$, almost all characteristics of the curve are same as of $y = \sin x$, but only different characteristics is

Range $= [-3, 3]$ | $[-1, 1]$, for $\sin x$

~~Maximum value occurs for $3\sin x = 1$~~

~~$\Rightarrow \sin x = 1/3$~~

~~$\Rightarrow x = n\pi + (-1)^n \sin^{-1}(1/3)$~~

~~$(n = 0, \pm 1, \pm 2)$~~

~~Minimum value occurs for $3\sin x = -1$~~

~~$\Rightarrow x = n\pi + (-1)^n \sin^{-1}(-1/3)$~~

In the graph, replace $\underline{1}$ by 3

Example 14 Draw the graph of $\boxed{y = \cos x}$.

Solution:- The following information regarding the equation is determined.

(1) Domain / Range

Domain $= (-\infty, \infty)$
Range $= [-1, 1]$

(2) Origin :- It does not pass through the origin ($\because 0 \neq \cos 0$)

(3) Symmetry:- It is symmetrical about y-axis
($\because y = \cos x = \cos(-x)$)

(4) Noticeable points
When $y = 0$, $\cos x = 0$ and so, $x = 2n\pi \pm \pi/2$
($n = 0, \pm 1, \pm 2, \ldots$)
Thus the graph cuts x-axis at
$x = \pm\pi/2, 5\pi/2, 3\pi/2$, etc. (Put $n = 0, 1, -1, 2, -2$)

(5) Period :-

Period $= 2\pi$ | Period of $\cos ax = \frac{2\pi}{|a|}$

(6) Maxima / minima

Maximum value is 1, which occur at points $x = 0, \pm 2\pi, \pm 4\pi, \ldots$

Minimum Value is -1, which occur at points $x = \pm\pi, 5\pi, 3\pi$ etc

Rough
$\cos x = 1 = \cos 0$ (max)
$\Rightarrow x = 2n\pi \pm 0$
($n = 0, \pm 1, \pm 2, \ldots$)

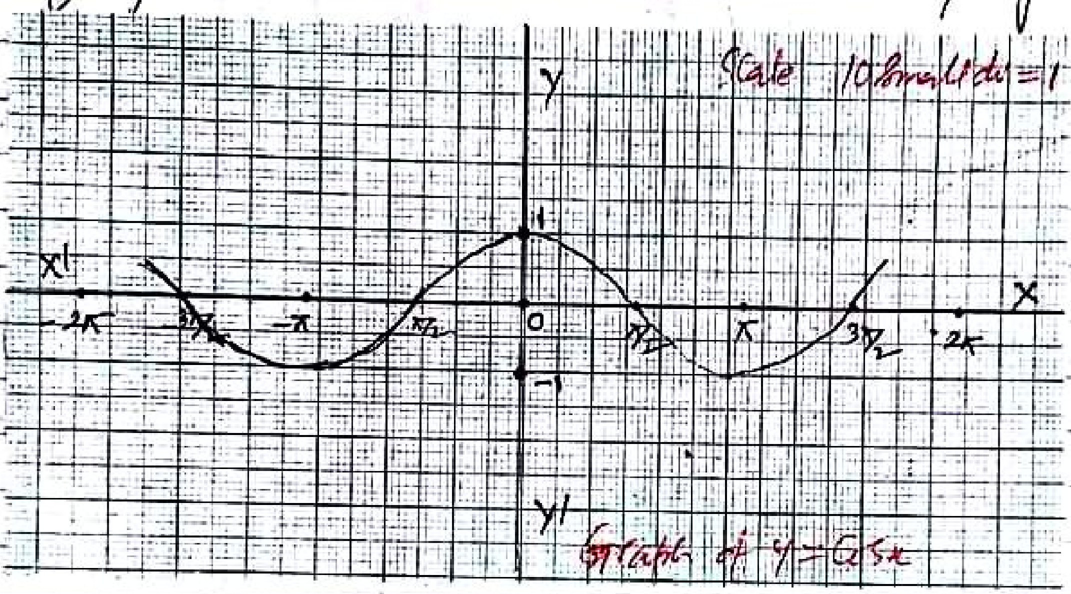
$\cos x = -1$
$= \cos \pi$
$\Rightarrow x = 2n\pi \pm \pi$, $n = 0, \pm 1, \ldots$

$\frac{dy}{dx} = -\sin x$ $\searrow$ <0 for 0 to $\pi/2$; $\searrow$ <0 for $\pi/2$ to π; $\nearrow$ >0 for π to $3\pi/2$; $\nearrow$ >0 for $3\pi/2$ to 2π π, $\pi/2$, $0, 2\pi$, $3\pi/2$ Rough

7. Increasing / decreasing.

y decreases from 0 to $\pi/2$ and increases from $\pi/2$ to π etc.

On the appeal of above mentioned characteristics, we can draw the graph of $y = \cos x$ as shown in the graph paper.



Exercise :- Sketch the graph of $y = 3\cos x$.

Example 15 : Draw the graph of $\boxed{y = \tan x}$

Solution :- The following information regarding the equation is noted.

1. Domain & Range

$$\text{Domain} = \mathbb{R} - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\right\}$$
$$\text{Range} = (-\infty, \infty)$$

2. Origin :- It passes through the origin ($0 = \tan 0$)

(3) Symmetry. (→ replaced by)

When $x \to -x$ and $y \to -y$, the eq^n is unaltered. So, the graph is symmetrical about the origin.

(4) Noticeable points

When $y = 0$, $\tan x = 0$

$\Rightarrow \tan x = \tan 0$

$\Rightarrow x = n\pi + 0$, $n = 0, \pm 1, \pm 2, \dots$

$\therefore$ the curve meets x-axis at $x = 0, \pm\pi, \pm 2\pi, \dots$

(5) Table for variation of y as x

x	-2π	$-\pi$	$-\pi/4$	0	$\pi/4$	π	2π
y	0	0	-1	0	1	0	0

(6) Period :-

Period $= \pi$ | Period of $\tan ax = \frac{\pi}{|a|}$

(7) (First derivative) $\frac{dy}{dx} = 1.\sec^2 x \geq 0$, so, the curve is increasing.

(8) Asymptotes

As $x \to \pm \pi/2, \pm 3\pi/2, \dots$, $y \to \infty$.

$\therefore$ the vertical asymptotes are

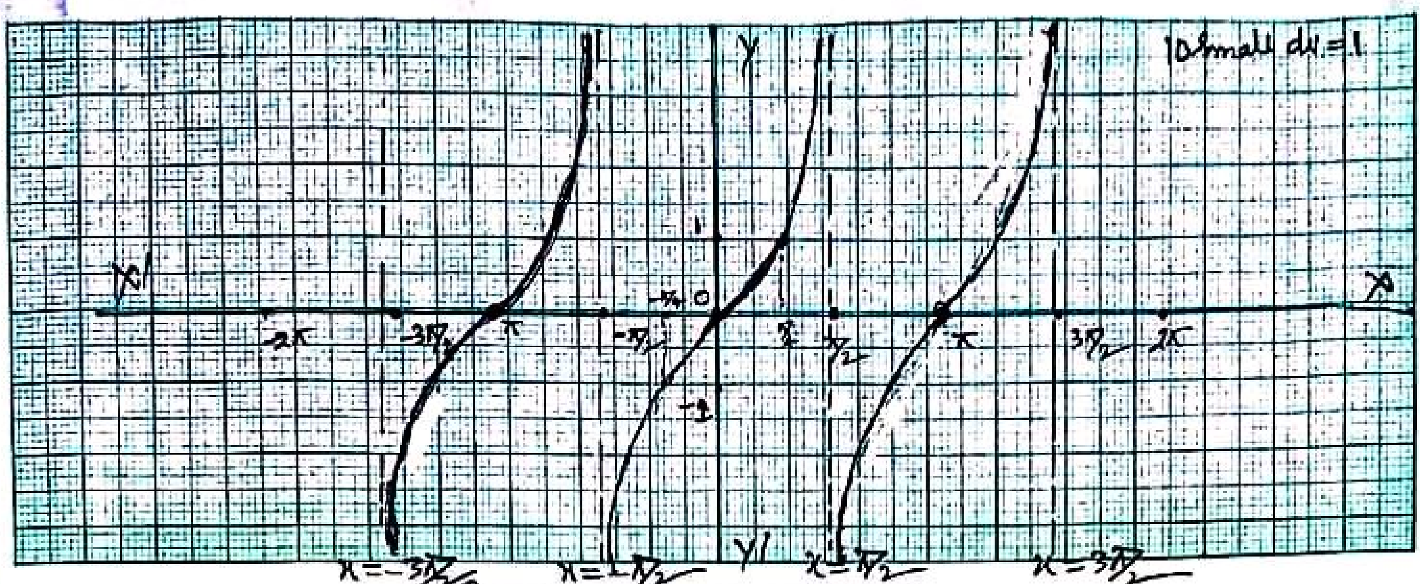
$x = \pi/2$, $x = -\pi/2$

$x = 3\pi/2$, $x = -3\pi/2$

$\dots$

$\dots$

Keeping in mind the information obtained above, we draw the graph of $y = \tan x$ as shown in the graph paper.



Graph of $y = \tan x$

Exercise :- $y = 3\tan x$

Example 16 (Exponential function)

Sketch the graph of $y = 2^x$

Solⁿ :- The following information regarding the equation is obtained

(1) (Domain / Range)

Domain $= (-\infty, \infty)$

Range $= (0, \infty)$ $\quad \because x = \log_2 y \Rightarrow y > 0$

(2) Origin : It does not pass through the origin $(\because 0 \neq 2^0)$

(3) Symmetry : not symmetrical

(4) Noticeable points :

when $x = 0$, $y = 1$. So the graph cuts the y-axis at $(0, 1)$.

(5) Table for the variation of y as x

x	-2	-1	0	1	2
y	0.25	0.5	1	2	4

(6) Increasing / Decreasing.

Since $y = 2^x$, as x increases, y also increases.

(7.) Asymptote

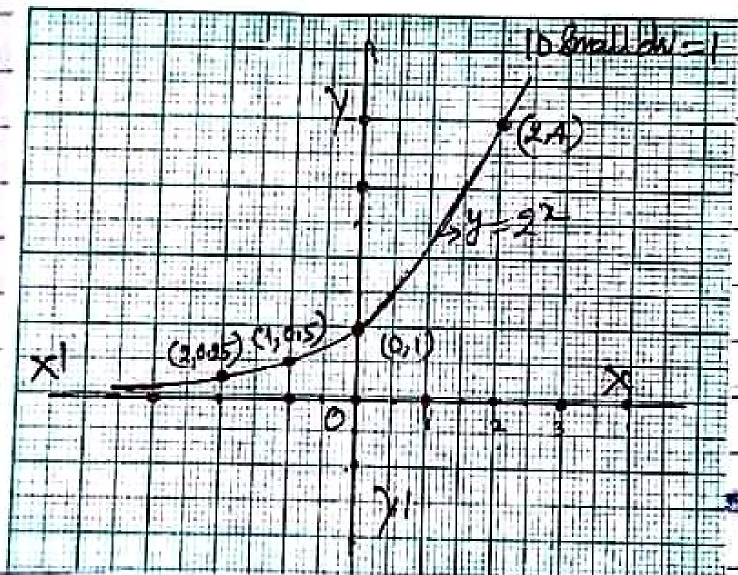
When $x \to -\infty$, $y \to 0$.

When $x = -\infty$, $y = 2^{-\infty} = \frac{1}{2^{\infty}} = 0$

$\therefore y = 0$ is an asymptote.

(8) Since $y = 2^x > 0$ always, no part of the curve lies below the x-axis.

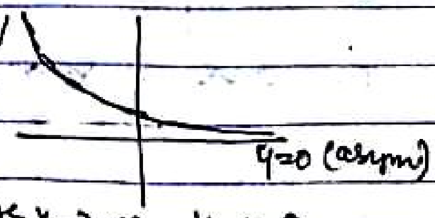
Thus, on the appeal of information obtained above, we can draw the graph as shown in the graph paper.



Exercise :- 1) $y = 3^x$

2) $y = 2^{-x}$

Hint: Graph



As $x \to \infty$, $y \to 0$

As x increases, y decreases.

Example 17 (logarithmic function).

Draw the graph of $\boxed{y = \log_2 x}$

Soln:- The characteristics of the graph are as under.

(1) Domain / range

($y = \log_2 x$ is defined only for $x > 0$)

Domain $= (0, \infty)$

Range $= (-\infty, \infty)$

(2) Origin :- It does not pass through the origin. ($\because 0 \neq \log_2 0$)

(3) Symmetry :- Not Symmetrical

(4) Noticeable points

When $y = 0$, $\log_2 x = 0$ so $x = 2^0 = 1$.
Hence, the graph meets x-axis at (1,0).

(5) Table for the variation of y as x

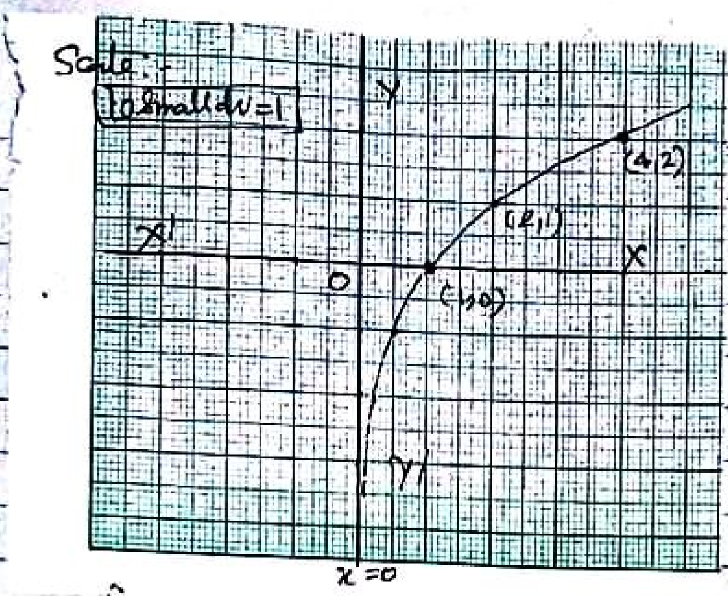
x		0.5	1	2	4
y		-1	0	1	2

(6) Asymptotes

when $x = 0$, $y = \infty$. $\{ \log_2 0 = \infty$

$\therefore$ $x = 0$ is an asymptote.

Keeping in mind the characteristics obtained above, we draw the graph as shown in the graph (aside)



Exercise $\boxed{y = \log_e x}$, $\boxed{y = \log_3 x}$ [ex 2.7]

$= \ln x$

Hint: Replace 2 by e in Example 17

Example 18 Trace the curve $\boxed{y = \frac{1}{x}}$, describing ~~the~~ its characteristics.

Solution :- The following information regarding the eqn is obtained.

1. Domain / Range

$$\text{Domain} = \mathbb{R} - \{0\}$$
$$\text{Range} = \mathbb{R} - \{0\}$$
$\left(\because x = \frac{1}{y} \text{ is defined only for } y \neq 0\right)$

2. Origin :- It does not pass through the origin

3. Symmetry : It is not Symmetrical about x-axis, y-axis and Symmetrical about the origin ($\because -y = \frac{1}{-x} \Rightarrow y = \frac{1}{x}$, no change)

5) Table showing variation of y as x

x		-2	-1	0.5	1	2
y		-0.5	-1	2	1	0.5

(6) Increasing / decreasing.

Since $y = \frac{1}{x}$, as x increases, y decreases.

(7) Asymptotes

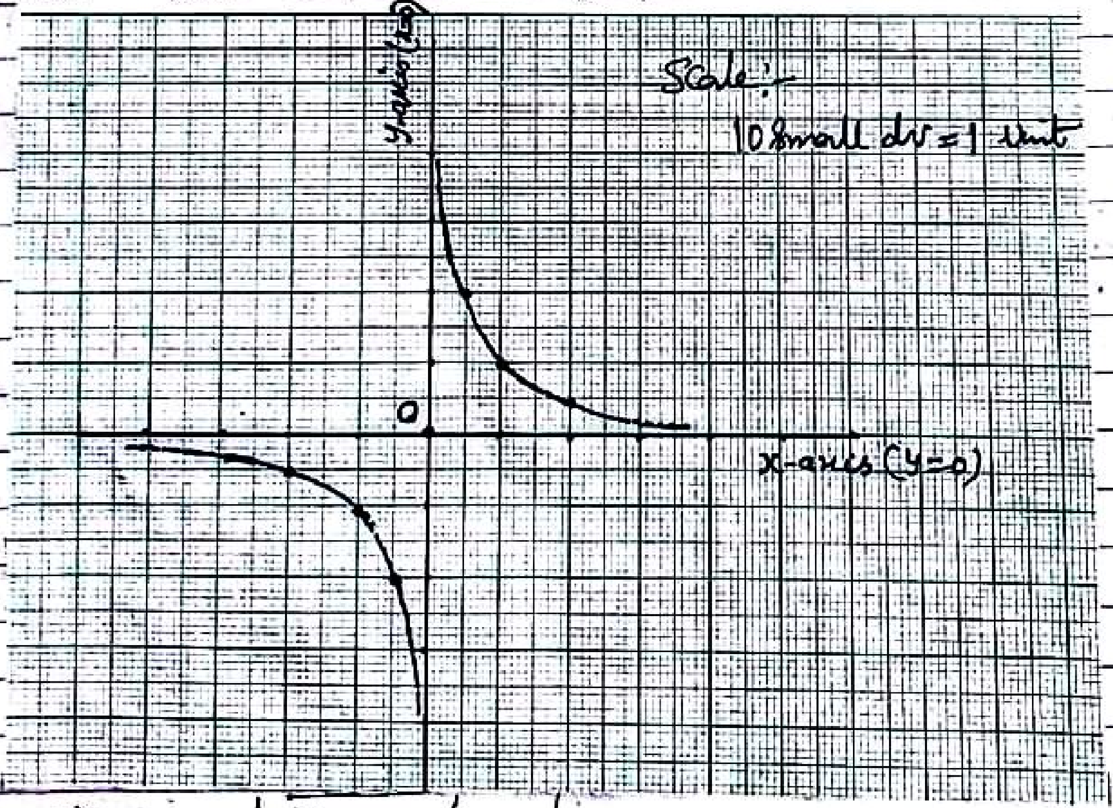
As $x \to 0$, $y \to \infty$

$\therefore$ $x = 0$ is the vertical asymptote.

As $x \to \infty$, $y \to 0$

$\therefore$ $y = 0$ is the horizonal asymptote.

Keeping on mind the information obtained above, we sketch the curve as shown on the graph paper.



Exercise:- $\boxed{y = \frac{1}{2x-1}}$ Hint: Replace x by $2x-1$

[End].